ON THE EQUATION $x^p = [(x^p - [x^p])N]$

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Abstract. We consider equation $x^p = [(x^p - [x^p])N]$ first with $p = 1/3$ and integer $N$, and then generalize results.

Introduction

In this paper, we study an equation

$$x^p = [(x^p - [x^p])N],$$

where $[x]$ is a floor function, the greatest integer $\leq x$.

First we consider a particular case of $p = \frac{1}{3}$, and integer $N$, and then proceed to generalize to an arbitrary real $p$ and $N$.

1. The equation $x = [(\sqrt[3]{x} - [\sqrt[3]{x}])N]$

We first consider the equation

$$x = \left(\sqrt[3]{x} - [\sqrt[3]{x}]\right)N,$$

where $[x]$ is the integer part of $x$, and $N$ is a positive integer.

As an example, for $N = 100$, the equation

$$x = \left(\sqrt[3]{x} - [\sqrt[3]{x}]\right)100$$

has one solution $x = 39$, with $\sqrt[3]{x} = 3.39121$.

1.1. Intermediate equation. To better understand, how the solutions to (1.1) arise, let’s consider an intermediate equation, without the outer square brackets:

$$x = (\sqrt[3]{x} - [\sqrt[3]{x}])N.$$

This equation has one solution for every interval $k^3 \leq x < (k + 1)^3$, where $(k + 1)^3 \leq N$. If $N \neq (k + 1)^3$ for some $k$, then the last is a cut-off interval $k^3 \leq x < N$, which may or may not have a solution. The Fig. (1) illustrates this.

The equation

$$x = (\sqrt[3]{x} - [\sqrt[3]{x}])100$$

has solutions

$$x_1 = 1.03125788101082, \quad x_2 = 9.14884844131658, \quad x_3 = 38.9362415949989.$$

The last solution of (1.4), $x = x_3 = 38.9362415949989$, is evidently related to the solution of (1.2), $x_0 = 39$. We see that $x_3$ differs from $x_1$ and $x_2$ by being close under an integer $x_0 = 39$. After some consideration, we can understand how it relates to $x_0$ being a solution to (1.2). We define $\Delta x = x_0 - x$, $\Delta y = (a + \frac{x}{N}) - (a + \frac{x_0}{N})$, where

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Figure 1. (Color online) Solutions of \( x = (\sqrt[3]{x} - \lfloor \sqrt[3]{x} \rfloor)N \). There are solutions for each interval \((k^3, (k+1)^3), k = 1, 2, 3\). Note closeness to an upper integer of the third root.

\[ a = \lfloor \sqrt[3]{x} \rfloor = \sqrt[3]{x} - \frac{x}{N}. \]

In order for \( x_0 \) to be a solution for (1.2), \( x \) must be such that increasing \( x \) by \( \Delta x \) will change \( \sqrt[3]{x} \) by at least \( \Delta y \), but less than by \( \Delta y + \frac{1}{N} \):

\[ \Delta y \leq (\sqrt[3]{x + \Delta x} - \sqrt[3]{x}) < \Delta y + \frac{1}{N}. \]

The left inequality of (1.6) is always satisfied. The right inequality may be approximated

\[ \frac{\Delta x}{3x^2} < \frac{\Delta x}{N} + \frac{1}{N}, \]

or

\[ \Delta x < \frac{1}{N \left( \frac{1}{3x^\frac{2}{3}} - 1 \right)} \]

1.2. Estimate of the density of solutions. With the condition (1.8), solution happens when lines \( x \) and \((\sqrt[3]{x} - \lfloor \sqrt[3]{x} \rfloor)N \) intersect within \( \Delta x \) of the next integer number, and \( \Delta x \) is the probability of such event. Summing for all approximately \( \sqrt[3]{N} \) intersections, we’ll get expectation of the number of solutions of the equation (1.1). For large \( N \) we can replace sum with integral. We get expected number of solutions

\[ n_{sol} = \int_{k=1}^{N^\frac{1}{3}} \frac{dk}{\frac{N}{3x^\frac{2}{3}} - 1} = \int_{k=1}^{N^\frac{1}{3}} \frac{dk}{\frac{N}{3x^2} - 1} \]

where \( k = \sqrt[3]{x} \). With \( N \) increasing to the infinity, the integral converges to 1.

The actual figures. For \( 0 < N \leq 12875 \) the numbers of solutions are

\[ \begin{array}{c|c}
0 & 4496, 1: 5054, 2: 2480, 3: 691, 4: 130, 5: 21, 6: 2, 7: 1, \\
\end{array} \]

which gives an average of 9.8888. For \( 0 < N \leq 25408 \), the average is 0.99067.

Fig. 2 shows a scatterplot of the actual number of solutions for given \( N \).

2. Generalization

2.1. Generalization of cubic root to a real power. Now let us consider a general equation (0.1)

\[ x = [(x^p - \lfloor x^p \rfloor)N], \]
ON THE EQUATION \( x^p = ([x^p] - [x^p])N \)

It is easy to see that our derivation remains the same, if we replace cubic root with an arbitrary real power \( p \), \( 0 < p < 1 \). The estimate for \( \Delta x \) becomes

\[
\Delta x < \frac{1}{N \frac{x^p}{x^{1-p}} - 1},
\]

and estimate of the number of solutions

\[
n_{sol} = \int_{k=1}^{N^p} \frac{dk}{N \frac{k^{1-p}}{x^{1-p}} - 1} = \int_{k=1}^{N^p} \frac{dk}{N \frac{k^{1-p}}{k^{1-p}} - 1}
\]

With \( N \) increasing to the infinity, the integral also converges to 1.

Computational check for \( p = \frac{1}{4} \), \( 0 < N \leq 15721 \), gives an average 0.98753.

2.2. **Generalization of \( N \) from integer to a real number.** Here again, most of the derivations of the previous section are applicable to a real \( N \), so that the formulas for the density of solutions remain the same. Moreover, changing \( N \) to \( N + \Delta N \), where \( 0 < \Delta N < 1 \), doesn’t change most of the solutions, since changing \( (x^p - [x^p])N \) to \( (x^p - [x^p])(N + \Delta N) \) changes the expression by about \( \frac{p}{2} \) on average, and we have \( p < 1 \).

**Conclusion.**

**References**